Diakoptic Theory for Multielement Antennas

GEORG GOUBAU, NARINDRA NATH PURI, SENIOR MEMBER, IEEE, AND FELIX K. SCHWERING, MEMBER, IEEE

Abstract—A theory is presented for the analysis of multielement antennas which consist of interconnected, conductive structure elements of electrically small dimensions. The theory is based on the retarded electromagnetic potentials which permit a diakoptic approach to the problem. The antenna is broken up into its individual structure elements. Each element is assumed to be excited by currents which are impressed at its terminals, i.e., junctions with adjacent elements (current coupling) and by the electric fields of the currents and charges on all the other elements (field coupling). Both excitations are treated independently. Each impressed current produces a "dominant" current distribution, a characteristic of the element, which can be readily computed. Current coupling is formulated by "intrinsic" impedance matrices which relate the scaler potentials at the terminals of an element, caused by its dominant current distributions, to the impressed currents of the element. Field coupling produces "scatter" currents on all the elements and is formulated by a "field-coupling" matrix which relates the scalar potentials at the terminals, caused by field coupling, to the impressed currents at all the terminals. Intrinsic and "field-coupling" matrices are combined to form the "complete" impedance matrix of the diakopted antenna. Enforcing continuity of the currents and equality of the scalar potentials at all the interconnections between the elements yields a system of linear equations for the junction currents and the input impedance of the antenna. Current coupling dominates field coupling. Field coupling is primarily affected by the dominant current distributions of the elements, and in general the scatter currents have negligible effect on it. Although detailed numerical investigations will be presented in another paper, a simple example is included here to demonstrate that the diakoptic theory yields very good results even if greatly simplified assumptions are made.

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G. Goubau was with Rutgers University, New Brunswick, NJ, before his death on October 17, 1980.

N. N. Puri is with the Department of Electrical Engineering, Rutgers University, New Brunswick, NJ 08903.

F. K. Schwering is with the U.S. Army Communication Electronics Command, Fort Monmouth, NJ 07703.

I. INTRODUCTION

An antenna which consists of a number of interconnected conductive structure elements of electrically small dimensions is shown in Fig. 1. This multielement design was chosen to obtain a broad-band highly efficient antenna of relatively low profile [1].

An analytical treatment of such a composite structure appears to be a rather hopeless undertaking. Commonly used numerical techniques would require computers with large storage capacity [2]. This paper offers a new approach to the problems of this kind, which holds promise for improved numerical efficiency. According to this approach the composite structure is diakopted into its individual structure elements. As a simple example Fig. 2 shows a diakopted dipole with end capacitor plates. Each structure element is characterized by electrical quantities which depend only on the size and shape of the element, and the assembly is treated similarly to the interconnection of n-port networks.

The excitation of each element is ascribed to two causes: a) the currents entering the element at its "terminals," i.e., junctions with adjacent elements or the source, and b) the fields of the currents and charges on all the other elements. The first is referred to as "current coupling" and the second as "field coupling." Each excitation is treated separately. Current coupling implies hypothetical sources with a single terminal and the capability of impressing a current onto a conductor. Although such sources violate the continuity condition, their assumption is permissible if the electromagnetic fields are expressed by the retarded electromagnetic potentials. Although the continuity condition is violated in the treatment of individual structure elements, it is restored when the elements are interconnected.

Let us for the moment disregard field coupling. If a current is impressed at a terminal of a structure element the current spreads over the surface of the element and produces a current

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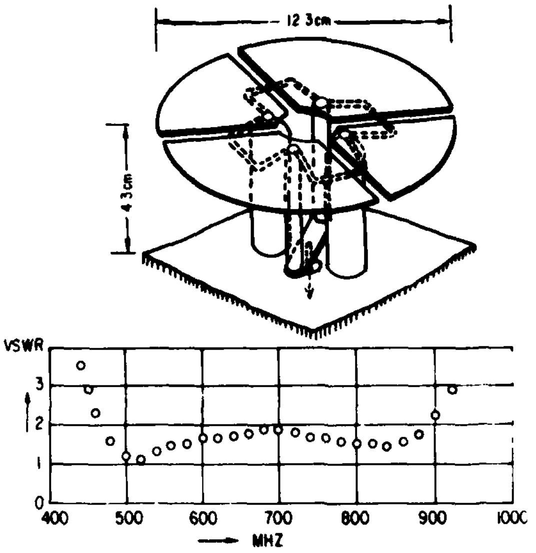

Fig. 1. Broad-band multielement antenna.

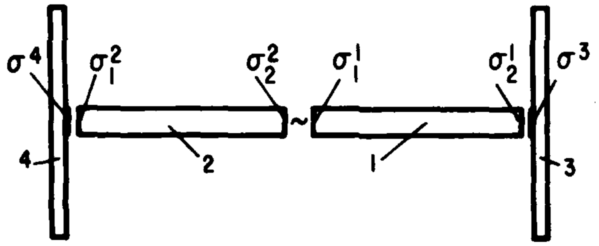

Fig. 2. Diakopted capacitively loaded dipole.

distribution which is uniquely determined by the geometry of the element and the location of the terminal. There are as many dominant current distributions as there are terminals. The relationship between the scalar potentials at the terminals, produced by the dominant current distributions, and the impressed currents is formulated by the "intrinsic impedance matrix" of the element.

Field coupling excites scatter currents which are superimposed on the dominant current distributions. The scalar potentials at the terminals due to field coupling depend on all the impressed currents. Their relationships with the impressed currents are formulated by a "field-coupling" matrix. Intrinsic impedance and "field-coupling" matrices, when combined, form the "complete impedance matrix" of the diakopted antenna. This "complete impedance matrix" relates the total scalar potentials to all the impressed currents.

Interconnection of the structure elements, which requires equal scalar potentials at the interconnected terminals and continuity of the junction currents, is formulated by an interconnection matrix. In this manner a system of linear equations is obtained which yields the junction currents and the input impedance of the antenna.

II. CURRENT COUPLING BETWEEN STRUCTURE ELEMENTS

A. Structure Elements with One Terminal

Consider one of the capacitor plates of the dipole in Fig. 2 separated from the other elements and suspended in space

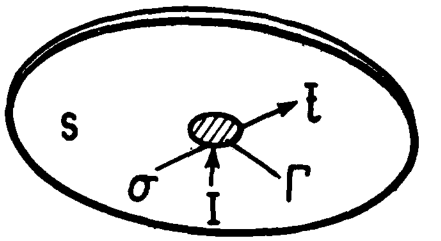

Fig. 3. Excitation of single terminal structure element.

with a current I impressed at the terminal, i.e., contact area in the center of the plate (Fig. 3). The contact area σ is considered very small compared with the surface of the element. Excitation by an impressed current cannot be treated with Maxwell's equations, because Maxwell's equations imply sources which *separate* positive and negative charges. In contrast, impressed currents require sources which *produce* charges. The regarded electromagnetic potentials do not impose any conditions on the source and therefore, can be used for our problem.

If $\bar{i}(\bar{r})$ is the surface current density and $q(\bar{r})$ the surface charge density due to the impressed current I, the retarded potentials are

$$\bar{A}(\bar{r}) = \frac{\mu}{4\pi} \int_S \bar{i}(\bar{r}')G(\bar{r}, \bar{r}')\, dS \qquad \text{(vector potential)} \qquad (1)$$

$$\phi(\bar{r}) = \frac{1}{4\pi\epsilon} \int_S q(\bar{r}')G(\bar{r}, \bar{r}')\, dS \qquad \text{(scalar potential)} \qquad (2)$$

with

$$G(\bar{r}, \bar{r}') = \frac{\exp(-jk|\bar{r} - \bar{r}'|)}{|\bar{r} - \bar{r}'|}, \qquad k = 2\pi/\lambda$$

where $\bar{r}'$ is the position vector of the charges and currents on the surface elements dS, and $\bar{r}$ is the point of observation. The quantities $\bar{i}(\bar{r})$ and $q(\bar{r})$ must satisfy the following two equations on the surface of the element outside the contact area σ:

$$\bar{E}(\bar{r}) \times d\bar{S} = -[j\omega\bar{A}(\bar{r}) + \bar{\nabla}\phi(\bar{r})] \times d\bar{S}$$

$$= \bar{0} \qquad \text{(boundary condition)} \qquad (3)$$

$$\bar{\nabla} \cdot \bar{i}(\bar{r}) + j\omega q(\bar{r}) = 0 \qquad \text{(continuity condition).} \qquad (4)$$

The condition that current flux through the boundary curve Γ of the contact area σ is the continuation of the impressed current I can be stated as

$$\oint_\Gamma \bar{t}(\bar{r}) \cdot \bar{i}(\bar{r})\, d\Gamma = I, \qquad (5)$$

where $\bar{t}(\bar{r})$ is a unit vector tangential to the surface S and normal to Γ. The current and charge distributions $\bar{i}(\bar{r})$ and $q(\bar{r})$ due to the impressed current I are termed as "dominant" distributions since the currents due to field coupling between the elements are, in general, relatively small. From the boundary condition (3)

$$\int_S \bar{E}(\bar{r}) \cdot \bar{i}(\bar{r})\, dS = -\int_S [j\omega \bar{A}(\bar{r}) + \bar{\nabla}\phi(\bar{r})] \cdot \bar{i}(\bar{r})\, dS = 0. \tag{6}$$

The surface of integration S is the surface of the element with the exclusion of the contact area. Using the relations

$$\begin{aligned} \bar{\nabla}\phi \cdot \bar{i}(\bar{r}) &= \bar{\nabla} \cdot (\phi(\bar{r})\bar{i}(\bar{r})) - \phi(\bar{r})\bar{\nabla} \cdot \bar{i}(\bar{r}) \\ &= \bar{\nabla} \cdot (\phi(\bar{r})\bar{i}(\bar{r})) + j\omega\phi(\bar{r})q(\bar{r}) \end{aligned} \tag{7}$$

and applying Gauss's theorem one obtains from (6)

$$\begin{aligned} j\omega \int_S [\bar{A}(\bar{r}) \cdot \bar{i}(\bar{r}) + \phi(\bar{r})q(\bar{r})]\, dS &= -\int_S \bar{\nabla} \cdot (\phi(\bar{r})\bar{i}(\bar{r}))\, dS \\ &= \oint_\Gamma \phi(\bar{r})\bar{i}(\bar{r}) \cdot \bar{t}(\bar{r})\, d\Gamma. \end{aligned} \tag{8}$$

If the contact area σ is sufficiently small ϕ can be considered constant within the contact area and on Γ. Thus with (5), (8) reduces to

$$j\omega \int_S [\bar{A}(\bar{r}) \cdot \bar{i}(\bar{r}) + \phi(\bar{r})q(\bar{r})]\, dS = \Phi I \tag{9}$$

where ϕ is the scalar potential at the contact area.

The ratio between Φ and I can be used to define an impedance which shall be termed "intrinsic impedance." If $\bar{A}$ and Φ are expressed by the current and charge distribution, the intrinsic impedance of the element is

$$\begin{aligned} Z &= \frac{\Phi}{I} \\ &= \frac{j\omega}{I^2} \int_S [\bar{A}(\bar{r}) \cdot \bar{i}(\bar{r}) + \phi(\bar{r})q(\bar{r})]\, dS \\ &= \frac{j\omega\mu}{4\pi} \int_S \int_S G(\bar{r}, \bar{r}') \\ &\quad \cdot \left[\frac{\bar{i}(\bar{r}) \cdot \bar{i}(\bar{r}')}{I^2} - \frac{1}{k^2} \frac{q(\bar{r})q(\bar{r}')}{Q^2} \right] dS'\, dS \end{aligned} \tag{10}$$

where $Q = I/j\omega$ is the total charge on the element. *The current and charge distribution functions $\bar{i}/I$ and q/Q are solely determined by the geometry of the element and the location of the coupling area.*

When the intrinsic impedance is computed with (10) for a conductor of any shape, for extremely low frequencies it takes the form

$$\underset{\omega \to 0}{Z} = \frac{1}{j\omega C} - \frac{1}{4\pi} \sqrt{\frac{\mu}{\epsilon}}, \tag{11}$$

where C is the static capacitance of the element.

Both terms in (11) result from the second term (charge term) in the integrand of (10). The first term in the integrand (current term) yields a lowest order contribution $\sim\omega$ and, therefore, vanishes for $\omega \to 0$. The first term $1/j\omega C$ in (11) is the one to be expected. The second term represents a negative resistance of $-30\ \Omega$ for a conductor in free space ($\eta = 120\pi$) and is not quite obvious. It is brought about by the fact that an impressed current produces a charge on the element without a countercharge, in contrast to a Maxwell source. If the scalar potential is expanded in a power series in ω one obtains

$$\begin{aligned} \phi(\bar{r}) &= \frac{1}{4\pi\epsilon} \int_S G(\bar{r}, \bar{r}')q(\bar{r}')\, dS' \\ &= \frac{1}{4\pi\epsilon} \left\{ \int_S \frac{q(\bar{r}')}{|\bar{r} - \bar{r}'|}\, dS' - jk \int_S q(\bar{r}')\, dS' + \cdots \right\}. \end{aligned}$$

The first term of this expansion is the static potential of the charges. The second term which is independent of $\bar{r}$ represents a potential, termed "background" potential ϕ_0, which is uniform in space and has no gradient. This means that ϕ_0 does not produce a field. It is this background potential which produces the $-30\ \Omega$ term in (11). When the element which we assumed to be suspended in space is within the antenna structure, the background potential is compensated because the total of the combined charges on all the other elements is equal, but opposite in sign, to the charge of the considered element. The background potential can be avoided if the retarded scalar potential is redefined as a modified potential $\hat{\phi}$:

$$\hat{\phi} = \phi - \phi_0 = \frac{1}{4\pi\epsilon} \int_S \hat{G}(\bar{r}, \bar{r}')q(\bar{r}')\, dS' \tag{12}$$

where

$$\hat{G}(\bar{r}, \bar{r}') = \frac{e^{-jk|\bar{r} - \bar{r}'|}}{|\bar{r} - \bar{r}'|} + jk. \tag{13}$$

This modified scalar potential will be used throughout the paper. It is preferred over the conventional potential ϕ since it will lead to expressions for the intrinsic impedance which are closer to physical expectation.

Use of $\hat{\phi}$ is legitimate as it does not conflict with Maxwell's theory. Since $\nabla\phi = \nabla\hat{\phi}$, the boundary condition (3) and the dominant current distribution derived from it remain unchanged if ϕ is replaced by $\hat{\phi}$. For a Maxwell system, ϕ and $\hat{\phi}$ are identical since $\int q\, dS$ extended over the surface of the entire structure is zero. The intrinsic impedance of a structure element with one terminal becomes

$$\begin{aligned} Z = \frac{\hat{\Phi}}{I} &= \frac{j\omega}{I^2} \int_S [\bar{A}(\bar{r}) \cdot \bar{i}(\bar{r}) + \hat{\phi}(\bar{r})q(\bar{r})]\, dS \\ &= \frac{j\omega\mu}{4\pi} \int_S \int_S \left[G(\bar{r}, \bar{r}') \frac{\bar{i}(\bar{r}) \cdot \bar{i}(\bar{r}')}{I^2} \right. \\ &\quad \left. - \frac{1}{k^2} \hat{G}(\bar{r}, \bar{r}') \frac{q(\bar{r})q(\bar{r}')}{Q^2} \right] dS'\, dS. \end{aligned} \tag{14}$$

Equation (14) represents a stationary formulation of the intrinsic impedance. This means small errors in the dominant current distribution have only a second-order effect on the intrinsic impedance (see Appendix I).

Excitation by an impressed current I at the terminal can be considered equivalent to excitation by an oscillating charge

$$Q = \frac{I}{j\omega} \tag{15}$$

which is placed above the contact area at a distance $d \to 0$ as shown in Fig. 4. The charge on the contact area σ consists essentially of the image charge $-Q$. As a result a charge $+Q$ is distributed over the surface area of the structure element because the net charge on the element must be zero. Under certain conditions charge excitation has certain practical advantages in the calculation of the dominant current distribution.

The intrinsic impedance Z of an element with one terminal can be represented by a lumped-element circuit as shown in Fig. 5. For low frequencies, i.e., when the dimensions of the element are small compared with the wavelength, C and L can be considered constant, while R increases proportionately with ω^2:

$$Z = \frac{1}{j\omega C} + j\omega L + R(\omega^2). \tag{16}$$

B. Structure Elements with Two or More Terminals

A structure element with two terminals such as the cylindrical conductors of Fig. 2 has two dominant current distributions; one associated with each of the independently impressed terminal currents (Fig. 6). Each dominant current distribution produces a scalar potential at both contact areas. If $\hat{\Phi}_{11}$ and $\hat{\Phi}_{21}$ are the potentials at the terminals 1 and 2 due to I_1, and $\hat{\Phi}_{12}, \hat{\Phi}_{22}$ those due to I_2, then the relationship between the total potentials $\hat{\Phi}_1$ and $\hat{\Phi}_2$ at the terminals and the impressed currents can be written as

$$\hat{\Phi}_1 = \hat{\Phi}_{11} + \hat{\Phi}_{12} = Z_{11}I_1 + Z_{12}I_2$$
$$\hat{\Phi}_2 = \hat{\Phi}_{21} + \hat{\Phi}_{22} = Z_{21}I_1 + Z_{22}I_2. \tag{17}$$

For a structure element with M terminals the relationship between the terminal potentials and the impressed currents is formulated by an $M \times M$ intrinsic-impedance matrix:

$$[\hat{\Phi}] = [Z][I] \tag{18}$$

where

$$Z_{jk} = \frac{j\omega}{I_j I_k} \int_S [\bar{A}_k(\bar{r}) \cdot \bar{i}_j(\bar{r}) + \hat{\phi}_k(\bar{r}) q_j(\bar{r})]\, dS \tag{19}$$

$$= \frac{j\omega\mu}{4\pi} \int_S \int_S \left[G(\bar{r}, \bar{r}') \frac{\bar{i}_j(\bar{r}) \cdot \bar{i}_k(\bar{r}')}{I_j I_k} - \frac{1}{k^2} \hat{G}(\bar{r}, \bar{r}') \frac{q_j(\bar{r}) q_k(\bar{r}')}{Q_j Q_k} \right] dS'\, dS, \tag{20}$$

$I_j = j\omega Q_j$ is the impressed current at the jth terminal (with $I_k = 0$) and $I_k = j\omega Q_k$ is the impressed current at the kth terminal (with $I_j = 0$). The quantities $\bar{i}_j$, q_j, $\bar{A}_j$, $\hat{\phi}_j$, and $\bar{i}_k$, q_k, A_k, $\hat{\phi}_k$ are the corresponding dominant currents distributions, dominant charge distributions, retarded vector, and scalar potentials, respectively. Equation (20) is derived in Appendix II. The symmetry of the intrinsic-impedance matrix $Z_{jk} = Z_{kj}$ is evident from (20). In Appendix I it is shown that (20) is a stationary representation of the impedance matrix elements.

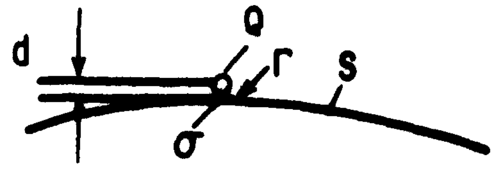

Fig. 4. Excitation of structure element by oscillating charge.

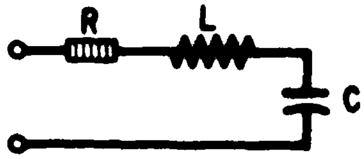

Fig. 5. Low frequency equivalent circuit for single terminal structure element.

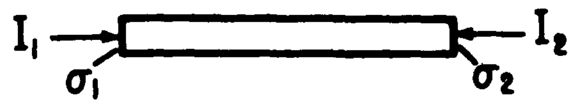

Fig. 6. Structure element with two terminals.

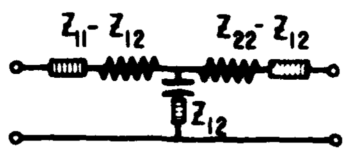

Fig. 7. Equivalent circuit for two-terminal structure element.

A lumped-element equivalent circuit for a structure element with two terminals is shown in Fig. 7. For sufficiently low frequencies the capacitors and inductors can be considered constant; while the resistors increase with ω^2.

A power series expansion of the right side of (20) in terms of $j\omega$ yields

$$Z_{jk} = \frac{1}{j\omega C} + j\omega L_{jk} + R_{jk}(\omega^2) + \cdots$$

where C is independent of j, k (in the low frequency limit the charge distribution on a given structure element is independent of the location of the contact area). L_{jk} and R_{jk} are positive for $j = k$, but in general are negative for $j \neq k$. In this case the term $j\omega L_{jk}$ may be counted as a higher order contribution to the capacitive term $1/j\omega C$. In Fig. 7 this assumption has been made and an inductor L_{12}, which should appear in series with the capacitor C, has been omitted. The resistor R_{12} of this circuit branch may be negative, but its magnitude will be smaller than that of the resistors $R_{11} - R_{12}$ and $R_{22} - R_{12}$ in the horizontal branches of the circuit.

III. FIELD COUPLING BETWEEN STRUCTURAL ELEMENTS

We now consider a diakopted structure and arbitrary currents impressed at the terminals. The capacitively loaded dipole of Fig. 2 may serve as an example. The terminals are identified by a superscript i and a subscript k; the superscript referring to the number of the element and the subscript referring to the number of the terminal on the element. If there were no field coupling between the elements, the current distributions on all the elements would be the dominant distributions associated with the impressed currents.

The field of a dominant current distribution is non-Maxwellian since the associated net charge is nonzero. If a current $I_k{}^i$ is impressed at the terminal $\binom{i}{k}$, the non-Maxwellian field of the dominant current and charge distribution $\bar{i}_k{}^i$, $q_k{}^i$ induces currents on all the other elements. The scatter fields excited by these induced currents are Maxwellian, since induced current distributions have no net charge. These "first-order" scatter fields excite second-order scatter fields and so on; each higher order having a greatly reduced amplitude. All these scatter fields when summed up form a multiple-scatter field which is Maxwellian. The currents and charges associated with the multiple-scatter field are distributed over all the surfaces S^n (including S^i) and shall be denoted $\delta\bar{i}_k{}^{in}$, $\delta q_k{}^{in}$; the superscripts and subscripts indicate that they are produced by the impressed current $I_k{}^i$ and located on the element n. In calculating $\delta\bar{i}_k{}^{in}$ and $\delta q_k{}^{in}$ the structure elements are assumed open-circuited at their terminals. Hence the total flux of the scatter currents through the rim of the contact areas is zero. Scatter currents do not contribute to the junction currents.

The total field generated by $I_k{}^i$ (all other terminals open-circuited) satisfies on every element the boundary conditions:

$$(j\omega(\bar{A}_k{}^i+\delta\bar{A}_k{}^i)+\bar{\nabla}(\mathring{\phi}_k{}^i+\delta\mathring{\phi}_k{}^i))\times d\bar{S}^n=\bar{0},$$
$$n=1,\cdots,i,\cdots,N \tag{21}$$

where $\bar{A}_k{}^i$, $\mathring{\phi}_k{}^i$ are the retarded potentials of the dominant current and charge distribution $\bar{i}_k{}^i$, $q_k{}^i$; and $\delta\bar{A}_k{}^i$, $\delta\mathring{\phi}_k{}^i$ are those of the scatter current and charge distributions $\delta\bar{i}_k{}^{in}$, $\delta q_k{}^{in}$ combined. N is the number of elements.

Since the electric field of $\bar{i}_k{}^i$, $q_k{}^i$ satisfies the boundary condition on S^i, it follows from (21) for $n=i$ that

$$(j\omega\delta\bar{A}_k{}^i+\bar{\nabla}\delta\mathring{\phi}_k{}^i)\times d\bar{S}^i=\bar{0}.$$

Thus,

$$\int_{S^i}[(j\omega\delta\bar{A}_k{}^i+\bar{\nabla}\delta\mathring{\phi}_k{}^i)\cdot\bar{i}_k{}^i]\,dS^i=0,$$
$$i=1,\cdots,N;\; k=1,\cdots,M_i. \tag{22}$$

Using the relations (7) and Gauss's theorem one obtains the "backscatter" potential due to the field interaction of the excited element with the other elements:

$$\mathring{\Phi}_{k,k}{}^{i,i}(F)I_k{}^i=j\omega\int_{S^i}(\delta\bar{A}_k{}^i\cdot\bar{i}_k{}^i+\delta\mathring{\phi}_k{}^iq_k{}^i)\,dS^i. \tag{23}$$

The letter F indicates field coupling; the first pair of indices $\binom{i}{k}$ refers to the terminal at which $\mathring{\Phi}$ is determined, and the second pair to the terminal of the impressed current which produces this potential.

As shown in Appendix III

$$\int_{S^i}(\delta\bar{A}_k{}^i\cdot\bar{i}_k{}^i+\delta\mathring{\phi}_k{}^iq_k{}^i)\,dS^i$$
$$=\sum_{n=1}^{N}\int_{S^n}(\bar{A}_k{}^i\cdot\delta\bar{i}_k{}^{in}+\mathring{\phi}_k{}^i\delta q_k{}^{in})\,dS^n. \tag{24}$$

Furthermore, from the boundary conditions (21), using the relations (7) and Gauss's theorem, follows

$$\int_{S^n}[(\bar{A}_k{}^i+\delta\bar{A}_k{}^i)\cdot\delta\bar{i}_k{}^{in}+(\mathring{\phi}_k{}^i+\delta\mathring{\phi}_k{}^i)\delta q_k{}^{in}]\,dS^n=0,$$
$$\text{for every } n \text{ including } i. \tag{25}$$

The right side of (25) is zero since for scatter currents the rim integral of Gauss's theorem in (5) is zero (see Appendix IV).

With (23), (24), and (25) one obtains the "backscatter" impedances

$$Z_{k,k}{}^{i,i}(F)=\frac{\mathring{\Phi}_{k,k}{}^{i,i}(F)}{I_k{}^i}$$
$$=-j\omega\left(\frac{1}{I_k{}^i}\right)^2\sum_{n=1}^{N}\int_{S^n}(\delta\bar{A}_k{}^i\cdot\delta\bar{i}_k{}^{in}$$
$$+\delta\mathring{\phi}_k{}^i\delta q_k{}^{in})\,dS^n, \tag{26}$$

which has to be added to the diagonal terms of the intrinsic-impedance matrix $Z_{k,k}{}^{i,i}$, using the notation of this section. Generalization of (26) to obtain the scatter field contributions to the off-diagonal terms is straightforward and yields

$$Z_{k,j}{}^{i,i}(F)=\frac{\mathring{\Phi}_{k,j}{}^{i,i}(F)}{I_j{}^i}$$
$$=-j\omega\frac{1}{I_k{}^iI_j{}^i}\sum_{n=1}^{N}\int_{S^n}(\delta\bar{A}_k{}^i\cdot\delta\bar{i}_j{}^{in}$$
$$+\delta\mathring{\phi}_k{}^i\delta q_j{}^{in})\,dS^n. \tag{27}$$

For $k=j$ (27) transforms into (26).

Let us now determine the potential $\mathring{\Phi}_{k,m}{}^{i,l}(F)$ produced at the terminal $\binom{i}{k}$ by the current $I_m{}^e$ impressed at the terminal $\binom{l}{m}$. From the boundary condition (21)

$$\int_{S^i}[j\omega(\bar{A}_m{}^l+\delta\bar{A}_m{}^l)+\bar{\nabla}(\mathring{\phi}_m{}^l+\delta\mathring{\phi}_m{}^l)]\cdot\bar{i}_k{}^i\,dS^i=0,$$
$$\tag{28}$$

and

$$\int_{S^n}[j\omega(\bar{A}_k{}^i+\delta\bar{A}_k{}^i)+\bar{\nabla}(\mathring{\phi}_k{}^i+\delta\mathring{\phi}_k{}^i)]\cdot\delta\bar{i}_m{}^{ln}\,dS^n=0,$$
$$\tag{29}$$

where the second equation holds for every n including i. The potentials $\delta\bar{A}_k{}^i$ and $\delta\mathring{\phi}_k{}^i$ characterize the scatter field which would be excited by $I_k{}^i$.

As before, we apply relations (7) and Gauss's theorem, and obtain from (28)

$$\mathring{\Phi}_{k,m}{}^{i,l}(F)I_k{}^i=j\omega\left[\int_{S^i}(\bar{A}_m{}^l\cdot\bar{i}_k{}^i+\mathring{\phi}_m{}^lq_k{}^i)\,dS^i\right.$$
$$\left.+\int_{S^i}(\delta\bar{A}_m{}^l\cdot\bar{i}_k{}^i+\delta\mathring{\phi}_m{}^lq_k{}^i)\,dS^i\right] \tag{30}$$

and from (29)

$$0 = \int_{S^n} (\bar{A}_k{}^i \cdot \delta\bar{i}_m{}^{ln} + \mathring{\phi}_k{}^i \delta q_m{}^{ln}) dS^n + \int_{S^n} (\delta\bar{A}_k{}^i \cdot \delta\bar{i}_m{}^{ln} + \delta\mathring{\phi}_k{}^i \delta q_m{}^{ln}) dS^n. \tag{31}$$

The first term in (30) represents the contribution to the terminal potential $\mathring{\Phi}_{k,m}{}^{i,l}(F)$ from the non-Maxwellian field of the dominant current and charge distribution $\bar{i}_m{}^l, q_m{}^l$, and the second term that from the scatter current and charge distributions $\delta\bar{i}_m{}^{ln}, \delta q_m{}^{ln}$.

As shown in Appendix III

$$\int_{S^i} (\delta\bar{A}_m{}^l \cdot \bar{i}_k{}^i + \delta\mathring{\phi}_m{}^l q_k{}^i) dS^i = \sum_{n=1}^{N} \int_{S^n} (\bar{A}_k{}^i \cdot \delta\bar{i}_m{}^{ln} + \mathring{\phi}_k{}^i \delta q_m{}^{ln}) dS^n. \tag{32}$$

Expressing $\mathring{\Phi}_{k,m}{}^{i,l}(F)$ in terms of an impedance

$$\mathring{\Phi}_{k,m}{}^{i,l}(F) = Z_{k,m}{}^{i,l}(F) I_m{}^l, \tag{33}$$

and using (30), (31), and (32), the field-coupling impedance between the terminals $({}^i_k)$ and $({}^l_m)$ becomes

$$Z_{k,m}{}^{i,l}(F) = \frac{1}{I_k{}^i I_m{}^l} j\omega \left[\int_{S^i} (\bar{A}_m{}^l \cdot \bar{i}_k{}^i + \mathring{\phi}_m{}^l q_k{}^i) dS^i - \sum_{n=1}^{N} \int_{S^n} (\delta\bar{A}_k{}^i \cdot \delta\bar{i}_m{}^{ln} + \delta\mathring{\phi}_k{}^i \delta q_m{}^{ln}) dS^n \right], \quad i \neq l. \tag{34}$$

Equations (27) and (34) yield the elements of the field-coupling impedance matrix $[Z(F)]$, which relates the scalar potentials $\mathring{\Phi}_k{}^i(F)$ at the terminals, caused by field coupling, to the impressed currents:

$$[\mathring{\Phi}(F)] = [Z(F)][I],$$

$$\mathring{\Phi}_k{}^i(F) = \sum_{l=1}^{N} \sum_{m=1}^{M_l} \mathring{\Phi}_{k,m}{}^{i,l}(F) = \sum_{l=1}^{N} \sum_{m=1}^{M_l} Z_{k,m}{}^{i,l}(F) I_m{}^l. \tag{35}$$

IV. COMPLETE IMPEDANCE MATRIX OF THE DIAKOPTED ANTENNA

The intrinsic-impedance matrices of the individual structure elements can be combined into a diagonal block impedance matrix $[Z(C)]$ by writing the matrix elements $Z_{k,j}$ (20) in the form $Z_{k,j}{}^{i,i}(C)$. The superscripts i identify the terminals k and j as belonging to the element i; the letter C indicates current coupling. The block matrix $[Z(C)]$, whose elements $Z_{k,j}{}^{i,l}(C)$ are zero for $i \neq l$, is the "current-coupling matrix" of the diakopted system and relates the terminal potentials

$$\mathring{\Phi}_k{}^i(C) = \sum_{j=1}^{M_i} Z_{k,j}{}^{i,i}(C) I_j{}^i$$

due to current coupling to the M_i impressed currents of the element i.

The sum of the matrices $[Z(C)]$ and $[Z(F)]$, i.e.,

$$[Z] = [Z(C)] + [Z(F)] \tag{36}$$

forms the "complete impedance matrix" of the diakopted antenna, which represents the relationship between the total terminal potentials

$$\mathring{\Phi}_k{}^i = \mathring{\Phi}_k{}^i(C) + \mathring{\Phi}_k{}^i(F) = \sum_{l=1}^{M} \sum_{m=1}^{M_l} \mathring{\Phi}_{k,m}{}^{i,l}$$

produced by current and field coupling to all impressed currents. In matrix form

$$[\mathring{\Phi}] = [Z][I]. \tag{37}$$

If the matrix elements $Z_{k,j}{}^{i,i}(C)$ (20) and $Z_{k,j}{}^{i,i}(F)$ (27) are added, the resulting elements $Z_{k,j}{}^{i,i}$ have the same form as those which pertain to field coupling between different structure elements (34). In other words (34) can be used as general formulation for all elements of the complete impedance matrix of the diakopted system, and the condition $i \neq l$ can be dropped.

The symmetry of the $[Z]$ matrix, i.e.,

$$Z_{k,m}{}^{i,l} = Z_{m,k}{}^{l,i} \tag{38}$$

can be easily verified, by expressing in (34) the vector and scalar potentials by the current and charge distributions according to (1) and (12).

Equation (34) represents a stationary formulation of the matrix elements of $[Z]$. This means first-order errors in the current and charge distributions lead to only second-order errors in the impedances (Appendix I).

Calculation of the radiation-coupling impedances according to (27) and (34) requires, in principle, computation of the scatter current and charge distributions. However numerical results obtained with this theory indicate that coupling by the scatter currents is a negligible effect. It has been found that coupling by the junction currents prevails over field coupling, and field coupling by the non-Maxwellian fields (radiated by the dominant current distribution) dominates over that by the Maxwellian fields (generated by the scatter currents). In principle field coupling effects by the scatter currents can be obtained with an iterative procedure, which is not discussed here.

If coupling by the Maxwellian scatter fields is neglected, the formula for the elements of the complete-impedance matrix for the diakopted system reduces to

$$Z_{k,m}{}^{i,l} = \frac{j\omega}{I_k{}^i I_m{}^l} \int_{S^i} (\bar{A}_m{}^l \cdot \bar{i}_k{}^i + \mathring{\phi}_m{}^l q_k{}^i) dS^i. \tag{39}$$

This means all the matrix elements can be computed from the dominant current distribution. It is worth mentioning that formulation of the antenna problem in terms of this approximate impedance matrix can also be derived in a more conventional manner by an application of the method of moments [3], [4]. For this purpose one would formulate the electric field integral equation of the entire (interconnected) antenna structure and reduce this equation to a linear system by employing the Galerkin version of the method of moments, while choosing the dominant current distributions of the various structure elements as (subsectional) basis and weight functions. After transformation of the linear system, by application of (7), into a matrix equation between the potentials and currents at the contact areas, the matrix coefficients are found to be identical with (39). The choice of the dominant current distributions as basis functions will alleviate the necessity for subdividing a given antenna into many small segments. Since each of these current distributions satisfies the conditioning $\bar{E}_{tan} = \bar{0}$ at its self-element (and since current coupling dominates) it can be expected that structure elements of comparatively large size will already yield accurate results. In other words the accent of the problem is shifted from solving an integral equation for the entire antenna to determination of the dominant current distributions of its structure elements which, in many cases, can be reduced to canonical problems. Moreover, because of the stationary properties of the matrix coefficients (39), the dominant current distributions have to be known only approximately in order to obtain impedance values of good accuracy. The numerical example discussed in Section VI confirms these predictions.

V. INTERCONNECTION OF THE STRUCTURE ELEMENTS AND INTERCONNECTION MATRIX

The requirement for the diakopted structure with impressed currents to be identical in performance with the assembled antenna are as follows.

a) The sum of the impressed currents is zero at every junction between the structure elements. This requirement assures that the Kirchoff condition of current continuity is satisfied and that the field of the assembled antenna is Maxwellian.
b) The scalar potentials at interconnected terminals are equal.
c) The potential difference between the input terminals is equated with the driving voltage of the antenna.

It should be emphasized that the continuity of current at a junction needs to be imposed only on the impressed currents which are in turn the fluxes of the dominant current distributions through the contact areas. The scatter currents do not contribute to the terminal currents since they are Maxwellian and have no net charge. Thus the satisfaction of boundary conditions (21) is consistent with the satisfaction of terminal condition.

Imposing these junction conditions the matrix equation (36) yields a system of linear equations for the unknown junction currents and the input impedance of the antenna. Using network theory concepts the reduction of (37) to this linear system of equations by enforcing the junction conditions can be formulated with a connection matrix $[C]$ which reduces the number of potentials and currents of the diakopted structure to those of the actual structure [5]. The potential-current

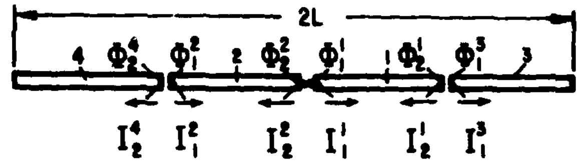

Fig. 8. Thin wire dipole treated as diakopted four-element system.

relationships are

$$[\dot{\Phi}] = [Z][I] \qquad \text{[diakopted antenna]} \tag{40}$$

$$[\dot{\Phi}]' = [Z]'[I]' \qquad \text{[assembled actual antenna]}. \tag{41}$$

Primed quantities refer to currents and potentials at various interconnections of the actual antenna. Requirements a), b), and c) represent Kirchoff's laws for interconnected structures and can be written as

$$[I] = [C][I]' \tag{42}$$

$$[\dot{\Phi}]' = [C]_t[\Phi] \tag{43}$$

$$[\dot{\Phi}]_t'[I]' = [\dot{\Phi}]_t[I]. \tag{44}$$

$[C]_t$ represents the transpose of $[C]$. Note that $\dot{\Phi}'$ denotes potential differences, i.e., voltages.

From (42), (43), and (44) the impedance of the actual interconnected structure can be written as

$$[Z]' = [C]_t[Z][C]. \tag{45}$$

The following example shows how $[C]$ and $[Z]'$ are obtained.

VI. EXAMPLE

As an example we apply the diakoptic theory to an ordinary thin-wire dipole antenna and compare the results with the data available in the literature. To obtain a multielement structure we cut each wire in half, as shown in Fig. 8, and consider each half as a structure element. The diakopted dipole is thus modeled by two structure elements with one terminal and two structure elements with two terminals, so that the total number of terminals is six. The complete impedance matrix of the diakopted structure $[Z]$ is therefore a 6 X 6 matrix. However there are only eight different impedances because the four structure elements have been assumed to be alike.

Using the enumerations of Fig. 8 the matrix equation (37) has the form

$$\begin{bmatrix} \dot{\Phi}_1^3 \\ \dot{\Phi}_2^1 \\ \dot{\Phi}_1^1 \\ \dot{\Phi}_2^2 \\ \dot{\Phi}_1^2 \\ \dot{\Phi}_2^4 \end{bmatrix} = \begin{bmatrix} Z_0 & Z_1 & Z_3 & Z_4 & Z_6 & Z_7 \\ Z_1 & Z_0 & Z_2 & Z_3 & Z_5 & Z_6 \\ Z_3 & Z_2 & Z_0 & Z_1 & Z_3 & Z_4 \\ Z_4 & Z_3 & Z_1 & Z_0 & Z_2 & Z_3 \\ Z_6 & Z_5 & Z_3 & Z_2 & Z_0 & Z_1 \\ Z_7 & Z_6 & Z_4 & Z_3 & Z_1 & Z_0 \end{bmatrix} \begin{bmatrix} I_1^3 \\ I_2^1 \\ I_1^1 \\ I_2^2 \\ I_1^2 \\ I_4^4 \end{bmatrix}$$

with

$$Z_0 = Z_{11}{}^{33} = Z_{22}{}^{11} = Z_{11}{}^{11} = Z_{22}{}^{22} = Z_{11}{}^{22} = Z_{22}{}^{44}$$
$$Z_1 = Z_{12}{}^{31} = Z_{21}{}^{13} = Z_{12}{}^{12} = Z_{21}{}^{21} = Z_{12}{}^{24} = Z_{21}{}^{42}$$
$$Z_2 = Z_{21}{}^{11} = Z_{12}{}^{11} = Z_{21}{}^{22} = Z_{12}{}^{22}$$
$$Z_3 = Z_{11}{}^{31} = Z_{11}{}^{13} = Z_{11}{}^{12} = Z_{11}{}^{21} = Z_{22}{}^{12} = Z_{22}{}^{21} = Z_{22}{}^{24} = Z_{22}{}^{42}$$
$$Z_4 = Z_{12}{}^{32} = Z_{21}{}^{23} = Z_{12}{}^{14} = Z_{21}{}^{41}$$
$$Z_5 = Z_{21}{}^{12} = Z_{12}{}^{21}$$
$$Z_6 = Z_{11}{}^{32} = Z_{11}{}^{23} = Z_{22}{}^{14} = Z_{22}{}^{41}$$
$$Z_7 = Z_{12}{}^{34} = Z_{21}{}^{43}$$

The darkened portion of the impedance matrix Z is the current-coupling matrix $Z(C)$.

The interconnection conditions require

$$I_1{}^3 = -I_2{}^1 = I_1 \qquad \hat{\Phi}_1{}^3 = \hat{\Phi}_2{}^1 = \Phi_1$$
$$I_2{}^2 = -I_1{}^1 = -I_0 \qquad \hat{\Phi}_1{}^1 - \hat{\Phi}_2{}^2 = V_0$$
$$I_2{}^4 = -I_1{}^2 = I_2 \qquad \hat{\Phi}_2{}^4 = \hat{\Phi}_1{}^2 = \Phi_2$$

where I_0, V_0 are input current and driving voltage of the antenna.

Because of the symmetry of the antenna

$$I_1 = -I_2; \Phi_1 = -\Phi_2; \hat{\Phi}_2{}^2 = -\hat{\Phi}_1{}^1.$$

Current and voltage matrix of the interconnected antenna are

$$[I]' = \begin{bmatrix} I_0 \\ I_1 \end{bmatrix}, \quad [\hat{\Phi}]' = \begin{bmatrix} V_0 \\ 0 \end{bmatrix}.$$

Thus the interconnection matrix becomes

$$[C]_t = \begin{array}{c} \\ (0) \\ (1) \end{array} \begin{array}{cccccc} \binom{3}{1} & \binom{1}{2} & \binom{1}{1} & \binom{2}{2} & \binom{2}{1} & \binom{4}{2} \\ \hline 0 & 0 & 1 & -1 & 0 & 0 \\ 1 & -1 & 0 & 0 & 1 & -1 \end{array},$$

and the impedance matrix of the assembled antenna is

$$[Z'] = 2 \begin{bmatrix} (Z_0 - Z_1) & (2Z_3 - Z_2 - Z_4) \\ (2Z_3 - Z_2 - Z_4) & (2Z_0 - 2Z_1 - Z_5 + 2Z_6 - Z_7) \end{bmatrix}$$

Hence

$$Z_{in} = \frac{V_0}{I_0} = (Z_0 - Z_1) - \frac{(2Z_3 - Z_2 - Z_4)^2}{2Z_0 - 2Z_1 - Z_5 + 2Z_6 - Z_7} \tag{46}$$

For the numerical calculation of the impedances the following simplifying assumptions have been made.

a) Coupling by scatter currents is negligible.

b) The dominant current distributions, which in this example are the same for all the elements, can be approximated by linear current distributions (uniform charge distribution).

Although the latter approximation is rather crude, one should expect reasonable results if the wire sections are short compared with the wavelength because all the impedance formulas are stationary expressions. Linear current distribution permits analytic formulations of all the impedances Z_0, Z_1, Z_2, etc., and numerical calculations with a pocket calculator (such as HP 25). The results obtained are presented in Fig. 9. The curves are plots (from a table by King [6]) of the real and the imaginary part of the input impedance of a dipole with $\ln(2l/\rho) = 5$ as a function of $\beta = kl$; $2l$ is the total length of the dipole and ρ is the wire radius. The crosses mark the values of the input impedance as obtained from (46) with the above assumptions. For $\beta = kl < 0.8$ the deviation of the real part of the input impedance from King's data is less than 10 percent, and for the imaginary part deviation is less than 1 percent. From this one can conclude that the linear approximation for the dominant current distribution is adequate if the length of a wire section is $<1/15\ \lambda$. This has been confirmed by computer results obtained with each dipole leg diakopted into four equal sections. These results are marked in Fig. 9 by dots and are in good agreement with King's (curves) even beyond the second resonance of the antenna.

Fig. 10 gives an indication of the convergence of numerical results as segmentation of a given antenna increases. A dipole antenna is assumed with each leg diakopted into N equal parts; the relative impedance errors $|\Delta R/R|$ and $|\Delta X/X|$ are plotted versus the antenna parameters $\Omega = 2 \ln(2l/\rho)$ for $\beta = 2$ and various N. The impedance errors were determined by comparison with King's results [6]. In general these errors decrease with increasing N. But the behavior of the curves in the region of smaller $\Omega(=10 \cdots 12)$ suggests that for any Ω there is an optimum value of N for which the errors are at a minimum,

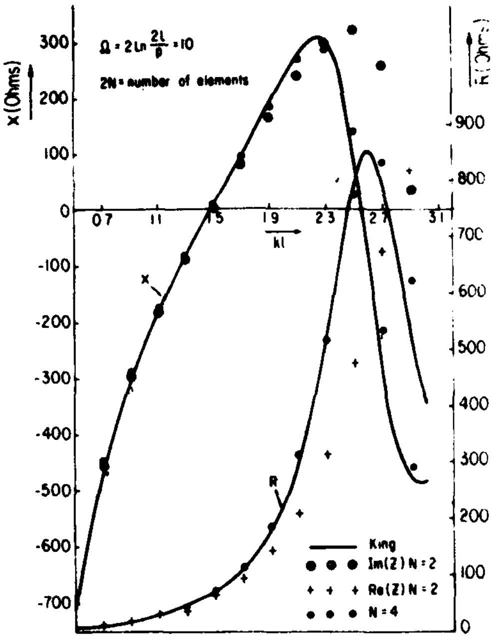

Fig. 9. Comparison of dipole impedance calculated with diakoptic theory versus King.

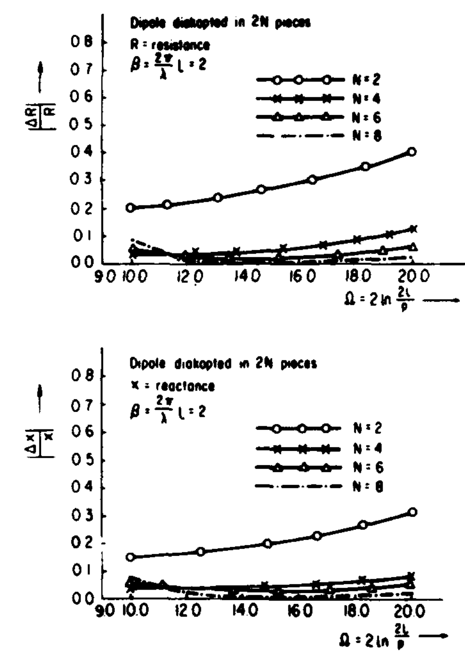

Fig. 10. Difference between King's results and diakoptic theory's results.

while a further increase in N would tend to reduce accuracy. As one should expect the optimum N increases with Ω, i.e., as the antenna becomes more slender. The length of the wire sections can be substantially increased if their dominant current distributions are approximated by sinusoidal rather than linear functions. Fig. 11 shows the input admittance of a folded dipole. The solid curves were taken from a table by King and Harrison [7]. The present method was evaluated counting the horizontal members of the antenna as one structure element each, while each vertical member was diakopted into two sections. For the dominant current distributions on these sections the sinusoidal approximation was used. Despite the comparatively large size of the vertical structure elements, computed impedance values were found to be in good accuracy up to and beyond the second resonance.

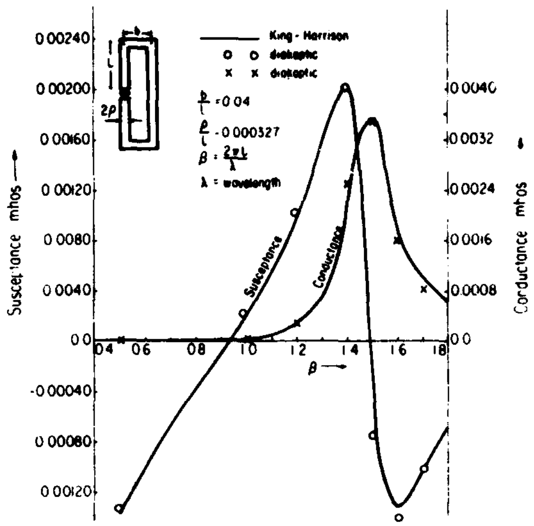

Fig. 11. Comparison of folded dipole admittance calculated with diakoptic theory versus King and Harrison.

VII. CONCLUSION

The major advantage of the diakoptic theory for multielement antennas, as discussed in this paper, is that the problem of determining the current distribution on the antenna need not be solved for the structure as a whole but only for the individual structure elements. Excitation of each structure element is ascribed to the currents at its junction with adjacent elements and to the fields of the surface currents on all the other elements. The current distributions produced by the junction currents have been termed dominant current distributions because they constitute the major portion of the currents on the composite antenna structure. The remainder of the currents are made up by scatter currents which are produced by field coupling. Field coupling, in first approximation, is determined by the dominant current distributions; while coupling by the scatter currents in general is negligible. Introduction of impedances for the characterization of structure elements and their interaction permits use of network theory concepts for the determination of the junction currents and the input impedance of the antenna. Formulation of all impedances by stationary expressions renders the results insensitive to computational errors in the current distributions. As demonstrated by the example given in the paper, even rather crude approximations to the dominant current distributions can yield good results.

APPENDIX I

Proof for the Stationary Formulation of the Impedances

1) To prove that

$$Z = \frac{j\omega}{I^2} \int_S (\bar{A} \cdot \dot{i} + \dot{\phi} q)\, dS \tag{A1}$$

represents a stationary formulation of the intrinsic impedance, we assume that the dominant current distribution $\bar{i}$ has an error of $\Delta\bar{i}$. The corresponding errors of q, $\bar{A}$, and $\hat{\phi}$ shall be denoted Δq, $\Delta\bar{A}$, and $\Delta\hat{\phi}$. Then

$$Z + \Delta Z = \frac{j\omega}{I^2} \int_S [(\bar{A} + \Delta\bar{A}) \cdot (\bar{i} + \Delta\bar{i}) + (\hat{\phi} + \Delta\hat{\phi})(q + \Delta q)]\, dS. \quad \text{(A2)}$$

The boundary condition for the correct dominant current distribution yields

$$\int_S [(j\omega\bar{A} + \bar{\nabla}\hat{\phi}) \cdot \Delta\bar{i}]\, dS = 0. \quad \text{(A3)}$$

Since the dominant current distribution is the continuation of the impressed current which is assumed to be unchanged, $\Delta\bar{i}$ is zero at the terminal, and (A3) can be written in the form

$$j\omega \int_S (\bar{A} \cdot \Delta\bar{i} + \hat{\phi}\Delta q)\, dS = 0. \quad \text{(A4)}$$

Using the relations

$$\int_S \bar{A} \cdot \Delta\bar{i}\, dS = \int_S \Delta\bar{A} \cdot \bar{i}\, dS$$
$$\int_S \hat{\phi}\Delta q\, dS = \int_S \Delta\hat{\phi} q\, dS \quad \text{(A5)}$$

along with (A4) one obtains

$$\int_S (\Delta\bar{A} \cdot \bar{i} + \Delta\hat{\phi} q)\, dS = \int_S (\bar{A} \cdot \Delta\bar{i} + \hat{\phi}\Delta q)\, dS = 0. \quad \text{(A6)}$$

Thus from (A1), (A3), and (A6)

$$\Delta Z = \frac{j\omega}{I^2} \int_S (\Delta\bar{A} \cdot \Delta\bar{i} + \Delta\hat{\phi}\Delta q)\, dS. \quad \text{(A7)}$$

This means ΔZ is of the second order.

2) In the case of a mutual-intrinsic impedance

$$Z_{jk} = \frac{j\omega}{I_k I_j} \int_S (\bar{A}_k \cdot \bar{i}_j + \hat{\phi}_k q_j)\, dS \quad \text{(A8)}$$

both the dominant current distributions $\bar{i}_k$ and $\bar{i}_j$ may have errors $\Delta\bar{i}_k$ and $\Delta\bar{i}_j$. Thus

$$\Delta Z_{jk} = \frac{j\omega}{I_k I_j} \int_S [(\Delta\bar{A}_k \cdot \bar{i}_j + \Delta\hat{\phi}_k q_j) + (\bar{A}_k \cdot \Delta\bar{i}_j + \hat{\phi}_k \Delta q_j) + (\bar{A}_k \cdot \Delta\bar{i}_j + \Delta\hat{\phi}_k \Delta q_j)]\, dS. \quad \text{(A9)}$$

Because the correct dominant current distributions satisfy the boundary condition $\bar{E} \times d\bar{S} = \bar{0}$,

$$\int_S (j\omega\bar{A}_k + \bar{\nabla}\hat{\phi}_k) \cdot \Delta\bar{i}_j\, dS = 0$$

or

$$j\omega \int_S (\bar{A}_k \cdot \Delta\bar{i}_j + \hat{\phi}_k q_j)\, dS = 0. \quad \text{(A10)}$$

Furthermore

$$\int_S (\Delta\bar{A}_k \cdot \bar{i}_j + \Delta\hat{\phi}_k q_j)\, dS = \int_S (\bar{A}_j \cdot \Delta\bar{i}_k + \hat{\phi}_j \Delta q_k)\, dS = 0. \quad \text{(A11)}$$

From (A10) and (A11), (A9) reduces to

$$\Delta Z_{jk} = \frac{j}{I_k I_j} \int_S (\Delta\bar{A}_k \cdot \Delta\bar{i}_j + \Delta\hat{\phi}_k \Delta q_j)\, dS. \quad \text{(A12)}$$

Thus ΔZ_{jk} is of second order.

3) To prove that (34) is a stationary expression for the field-coupling impedances we treat the assembly of disconnected structure elements like a single body. This means when a current is impressed on terminal $\binom{l}{k}$ we consider the dominant current distribution $\bar{i}_k{}^l$ together with the associated scatter currents $\delta\bar{i}_k{}^l$ which are distributed over all the elements as a dominant current distribution of the system. The coupling impedances between any two terminals can then be formulated like mutual-intrinsic impedances (A8),

$$Z_{k,m}{}^{l,l}(F) = \frac{j\omega}{I_k{}^l I_m{}^l} \int_{\Sigma S^n} [(\bar{A}_m{}^l + \delta\bar{A}_m{}^l) \cdot (\bar{i}_k{}^l + \delta\bar{i}_k{}^l) + (\hat{\phi}_m{}^l + \delta\hat{\phi}_m{}^l) \cdot (q_k{}^l + \delta q_k{}^l)]\, dS. \quad \text{(A13)}$$

The error $\Delta Z_{km}{}^{ll}$ produced by errors in the current distributions $\bar{i}_k{}^l$, $\delta\bar{i}_k{}^l$ and $\bar{i}_m{}^l$, $\delta\bar{i}_m{}^l$ is obtained from (A12),

$$\Delta Z_{k,m}{}^{l,l}(F) = \frac{j\omega}{I_k{}^l I_m{}^l} \int_{\Sigma S^n} [(\Delta\bar{A}_k{}^l + \Delta\delta\bar{A}_k{}^l) \cdot (\Delta\bar{i}_m{}^l + \Delta\delta\bar{i}_m{}^l) + (\Delta\hat{\phi}_k{}^l + \Delta\delta\hat{\phi}_k{}^l) \cdot (\Delta q_m{}^l + \Delta\delta q_m{}^l)]\, dS, \quad \text{(A14)}$$

and is of second order. This relation can also be derived from (34), but only in a rather cumbersome manner.

APPENDIX II

Derivation of (19)

Consider a structure element with several terminals and let k and j be any two terminals where currents I_k and I_j are impressed. The corresponding dominant current and charge distributions $\bar{i}_k$, q_k and $\bar{i}_j$, q_j produce the fields $\bar{E}_k$ and $\bar{E}_j$ which satisfy the boundary conditions

$$\bar{E}_k \times d\bar{S} = -(j\omega\bar{A}_k + \bar{\nabla}\hat{\phi}_k) \times d\bar{S} = \bar{0} \quad \text{(A15)}$$

$$\bar{E}_j \times d\bar{S} = -(j\omega\bar{A}_j + \bar{\nabla}\hat{\phi}_j) \times d\bar{S} = \bar{0}. \tag{A16}$$

Since the currents $\bar{i}_k$ and $\bar{i}_j$ are tangential to the surfaces it follows from the boundary conditions

$$\int_S (j\omega\bar{A}_k + \bar{\nabla}\hat{\phi}_k) \cdot \bar{i}_k \, dS = 0 \tag{A17}$$

$$\int_S (j\omega\bar{A}_k + \bar{\nabla}\hat{\phi}_k) \cdot \bar{i}_j \, dS = 0. \tag{A18}$$

Using the vector identity

$$\bar{\nabla} \cdot (\hat{\phi}\bar{i}) = \bar{\nabla}\hat{\phi} \cdot \bar{i} + \hat{\phi}(\bar{\nabla} \cdot \bar{i}),$$

$$\text{with } \bar{\nabla} \cdot \bar{i} = -j\omega q \tag{A19}$$

and applying Gauss's theorem as in (8) of Section II, (A17) and (A18) can be written in the form

$$-\int_S [\bar{\nabla} \cdot (\hat{\phi}_k \bar{i}_k)] \, dS = \hat{\phi}_{kk} I_k$$

$$= j\omega \int_S (\bar{A}_k \cdot \bar{i}_k + \hat{\phi}_k q_k) \, dS \tag{A20}$$

$$-\int_S [\bar{\nabla} \cdot (\hat{\phi}_k \bar{i}_j)] \, dS = \hat{\Phi}_{jk} I_j$$

$$= j\omega \int_S (\bar{A}_k \cdot \bar{i}_j + \hat{\phi}_k q_j) \, dS, \tag{A21}$$

where $\hat{\Phi}_{kk}$ is the potential at the terminal k due to $\bar{i}_k$ and q_k and $\hat{\Phi}_{jk}$ that at the terminal j due to $\bar{i}_k$ and q_k. Of course $\bar{i}_k$ and q_k are induced on the structure by I_k at the kth terminal with all the other terminals open-circuited, i.e., with other impressed currents set equal to zero. For $j = k$, (A21) transforms into (A20).

With

$$\hat{\Phi}_{jk} = Z_{jk} I_k \tag{A22}$$

one obtains from (A21) the expression for Z_{jk} given in (19). The expression in (20) is obtained if the potentials $\bar{A}_k$ and $\hat{\phi}_k$ are expressed by (1) and (12), respectively.

APPENDIX III

Derivation of (23)

Let $\bar{i}_k{}^l$ be a dominant current distribution on the surface S^l and $\delta\bar{i}_k{}^{ln}$ be the scatter current distribution on S^n produced by $\bar{i}_k{}^l$ ($n = 1, 2, \cdots, N$).

$$\delta\bar{A}_k{}^l(\bar{r}) = \frac{\mu}{4\pi} \sum_{n=1}^{N} \int_{S^n} \delta\bar{i}_k{}^{ln}(\bar{r}') G(\bar{r}, \bar{r}') \, dS^n(\bar{r}'). \tag{A23}$$

Multiplying (A23) with $\bar{i}_k{}^l(\bar{r})$ and integrating over S^l

$$\int_{S^l} \delta\bar{A}_k{}^l(\bar{r}) \cdot \bar{i}_k{}^l(\bar{r}) dS^l$$

$$= \frac{\mu}{4\pi} \int_{S^l} \bar{i}_k{}^l(\bar{r}) \cdot \sum_{n=1}^{N} \int_{S^n} \delta\bar{i}_k{}^{ln}(\bar{r}') G(\bar{r}, \bar{r}') \, dS^n(\bar{r}') \, dS^l(\bar{r})$$

$$= \frac{\mu}{4\pi} \sum_{n=1}^{N} \int_{S^n} \delta\bar{i}_k{}^{ln}(\bar{r}') \int_{S^l} \bar{i}_k{}^l(\bar{r}) G(\bar{r}, \bar{r}') \, dS^l(\bar{r}) \, dS^n(\bar{r}')$$

$$= \sum_{n=1}^{N} \int_{S^n} \delta\bar{i}_k{}^{ln}(\bar{r}) \cdot \bar{A}_k{}^l(\bar{r}) \, dS^n;$$

$$G(\bar{r}, \bar{r}') = G(\bar{r}', \bar{r}). \tag{A24}$$

Similarily it can be shown that

$$\int_{S^l} \delta\hat{\phi}_k{}^l q_k{}^l \, dS^l = \sum_{n=1}^{N} \int_{S^n} \delta q_k{}^{ln} \hat{\phi}_k{}^l \, dS^n. \tag{A25}$$

The proof for (32) in the body of the paper follows the same outline as given above.

APPENDIX IV

If a current is impressed on any terminal of a diakopted structure there will be capacitive currents between the contact areas of the disconnected elements which have not been considered in the derivation of the field coupling impedances. One might therefore conclude that the formulas are approximations which require the gaps between adjacent contact areas to be so large that capacitive currents are negligible. The purpose of this appendix is to show that the expressions for $Z_{k,k}{}^{i,i}(F)$ and $Z_{k,m}{}^{i,l}(F)$ are correct even if the gaps are infinitely small.

Fig. 12 shows two structure elements: a cylindrical rod 1 and a disc 2 with the opposing contact areas $\sigma_2{}^1$ and $\sigma_1{}^2$. If a current is impressed on the terminal $\binom{1}{1}$ of the rod there will be a potential difference between $\sigma_2{}^1$ and $\sigma_1{}^2$; which, in turn, produces a displacement current between these terminals. The potential difference which is the line integral of the electric potential field between $\sigma_2{}^1$ and $\sigma_1{}^2$ is essentially determined by the charges on the contact areas. If the gap is made smaller and smaller the potential difference approaches zero and the total current distribution becomes the dominant current distribution of the interconnected elements. Because of the equivalence of current and charge excitation displacement currents at contact areas are equivalent to impressed currents. Thus the situation discussed above is the excitation of a diakopted structure, not by one, but by three impressed currents. To produce excitation by one impressed current in accordance with our theory, the displacement currents must be compensated so that there is no current flux from the contact area onto the surface S of the element (S by definition does not contain the contact areas of the element). The magnitude of these compensating currents does not enter into the analysis because if the impressed currents of the diakopted structure are identical with the junction currents of the inter-

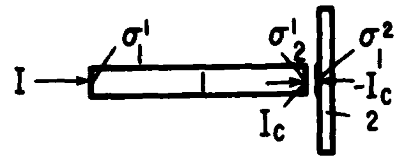

Fig. 12. Compensation of capacitive currents at contact areas.

connected structure there are no displacement currents between adjacent contact areas and the sum of all the compensating currents is zero.

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REFERENCES

[1] G. Goubau and F. Schwering, "Proceedings of the ECOM-ARO workshop on electrically small antennas," Fort Monmouth, NJ, pp. 63–67, Oct. 1976.

[2] K. R. Demarest and R. J. Garbacz, "Anomalous behavior of near fields calculated by method of moments," *IEEE Trans. Antennas Propagat.*, vol. AP-27, pp. 609–615, Sept. 1979.

[3] R. F. Harrington, *Field Computation by Moment Method*. New York: Macmillan, 1968, ch. 1.

[4] G. A. Thiele, "Wire antennas," in *Computer Techniques for Electromagnetics*, R. Mittra, Ed. New York: Pergamon, 1973, ch. 2.

[5] G. Kron, *Tensors for Circuits*. New York: Dover.

[6] R. W. P. King, "Cylindrical antennas and arrays," in *Antenna Theory*, R. E. Collins and F. J. Zucker, Eds. New York: McGraw-Hill.

[7] R. W. P. King and C. W. Harrison, *Antennas and Waves, A Modern Approach*. Cambridge, MA: M.I.T., 1969.

Georg Goubau was born in Munich, Germany, on November 29, 1906. He studied physics at the Technical University of Munich where he received the M.A. degree in 1931, the Ph.D. degree in 1932, and the Dr. habilitatus degree in 1936.

From 1931 to 1939 he was engaged in research and teaching at the Technical University of Munich. In 1939 he was appointed Professor and Director of the Department of Applied Physics at the University of Jena, now in East Germany. From 1947 until his retirement in 1973 he worked at the U.S. Army Electronics Command Laboratories in Fort Monmouth, NJ. In 1974 he joined the Faculty of the Electrical Engineering Department of Rutgers University as a Visiting Professor. His early contributions were in the field of ionospheric research; but later his major interests turned to microwave theory and techniques, free space and guided propagation of electromagnetic and optical radiation, and antennas, in particular those of electrically small configuration. He coauthored and edited a book on electromagnetic waveguides and resonators which has been published in German and English.

Dr. Goubau received the Harry Diamont Memorial Award of the IRE for his basic contributions to the theory of surface waves and the invention of the surface wave transmission line in 1957. He was recipient of the 1961 John T. Bolljahn Award of the IRE Professional Group on Antennas and Propagation for his paper on the theory of beam waveguides. In 1972 he received the Decoration for Meritorious Civilian Service of the Department of the Army. He was a member of Sigma Xi and U.S. Commission VI of URSI.

Dr. Goubau died on October 17, 1980.

Narindra Nath Puri (M'64–SM'79) was born in New Delhi, India, on November 30, 1933. He received the bachelor's degree in electrical engineering in 1954 and the Ph.D. degree from the University of Pennsylvania's Moore School of Electrical Engineering, where he was a Harrison Research Fellow.

He came to the United States in 1957 after working for two years with Brown Boveri and Cie in Germany and Switzerland. He then spent a year as a Research Assistant at the University of Wisconsin. In 1960 he joined the Drexel Institute of Technology and the University of Pennsylvania. In 1963 he becamse a full-time Associate Professor at Drexel Institute but gave up this position in 1964 when he joined the staff of the General Electric Missile and Space Division as Manager of Guidance and Control Subsystem Analysis. Since 1968 he has been at Rutgers University as Professor of Electrical Engineering. His consultation with RCA involved working on control of satellites with long flexible booms and with Messerschmidt-Bolkow-Blohm of Germany in the area of dual spin diagnosis and mathematical modeling of multi-element antennas.

Felix K. Schwering (M'60) was born in Cologne, Germany, on June 4, 1930. He received the Dipl. Ing. degree in electrical engineering and the Ph.D. degree from the Technical University of Aachen, Aachen, Germany, in 1954 and 1957, respectively.

From 1956 to 1958, he was an Assistant Professor at the Technical University of Aachen. In 1958 he joined the U.S. Army Research and Development Laboratory in Fort Monmouth, NJ, where he performed basic research in free space and guided propagation of electromagnetic waves. From 1961 to 1964 he worked as a Member of the Research Staff of the Telefunken Company, Ulm, Germany, on radar propagation studies and missile electronics. In 1964 he returned to the U.S. Army Electronics Command, Fort Monmouth, NJ, and has since been active in the fields of electromagnetic-wave propagation, diffraction and scatter theory, theoretical optics, and antenna theory. Recently, he has been involved in particular in millimeter-wave antenna and propagation studies. He is a Visiting Professor at Rutgers University.

Dr. Schwering is a member of Sigma Xi and is a recipient of the 1961 Best Paper Award of the IRE Professional Group on Antennas and Propagation (jointly with G. Goubau).